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Lie symmetries of a generalised non-linear Schrödinger equation: III. Reductions to third-order ordinary differential equations

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Abstract. The study of group invariant solutions of the generalised non-linear Schrödinger equation (GNLSE) is continued. It is shown that eight types of subgroups of the symmetry group lead, via symmetry reduction, to third-order real ordinary differential equations, giving both the phase and the absolute value of the solution. Only two of the reductions provide a Painlevé-type equation and both of them only for the cubic GNLSE. This equation is solved in terms of the fourth Painlevé transcendent.

1. Introduction

This paper is the third in a series devoted to the construction of group invariant solutions of a generalised non-linear Schrödinger equation (GNLSE), namely the cubic-quintic equation

$$\mathrm{i}\psi_t + \Delta\psi = a_0\psi + a_1|\psi|^2\psi + a_2|\psi|^4\psi \tag{1.1}$$

$$\psi = \psi(x, y, z, t) \in \mathbb{C} \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$(a_1, a_2) \neq (0, 0) \qquad a_i \in \mathbb{R} \qquad a_i = \text{constant} \qquad i = 0, 1, 2.$$

The Lie group of local point symmetries of equation (1.1) was determined in the first paper of the series [1] (hereafter referred to as I). For $a_1 a_2 \neq 0$ this group is the extended Galilei group. For $a_1 = 0$, or $a_2 = 0$ (not simultaneously) the group is the extended Galilei-similitude group, containing dilations, in addition to translations, rotations, proper Galilei transformations and constant changes of phase. All subalgebras of the Galilei algebra and the Galilei-similitude algebra were classified into conjugacy classes in I. The second paper in the series [2] (hereafter referred to as II) was devoted to symmetry reductions and to a search for exact analytic group invariant solutions. More specifically, all reductions to partial differential equations in two variables were presented, as well as all reductions to algebraic equations. Reductions to ordinary differential equations (ODE) were also presented, with the restriction that these be of first or second order. All algebraic equations and first-order ODE were solved, as were all second-order ODE that have the Painlevé property [3-5] (no moving singularities, other than poles, i.e. no critical points, the position or character of which depends on the initial conditions).

The physical motivation for our interest in the GNLSE (1.1) was also presented in I and II. Here we continue and complete the study of ODE resulting from a symmetry reduction of (1.1), namely we obtain and discuss all third-order ODE that provide group invariant solutions.

The fact that we are interested, in this paper, in group invariant solutions and in reductions to ODE, imposes two important restrictions on the subgroups to be considered.

(i) The generic orbits of the subgroups, when acting on the space $X \times U$ of independent (x, y, z, t) and dependent (ψ, ψ^*) variables must have orbits of codimension $k=3$.

(ii) The corresponding three invariants of the group action on $X \times U$, say $I_i(x, y, z, t, \psi, \psi^*)$, $i=1,2,3$, must provide an invertible mapping from the space of dependent variables to that of the invariants.

When these two conditions are satisfied, we can express the solution ψ (and its complex conjugate ψ^*) in terms of the invariants and then obtain group invariant solutions. The condition on the codimension assures that we reduce to an ODE. In view of the type of action that the Galilei-similitude group has on $X \times U$, the reduction formula will always be

$$\psi(x, y, z, t) = \alpha(x, y, t) f(\xi) \qquad \xi = \xi(x, y, z, t) \tag{1.2}$$

where α and ξ are known functions. The actual form of these functions is determined by the choice of subgroup.

Substituting (1.2) into the GNLSE (1.1) we obtain a complex ODE for the function $f(\xi)$. If the action of the subgroup H, restricted to the space of spacelike variables $\{x, y, z\}$, is transitive the ODE is of first order (and has been treated exhaustively in II). Otherwise, the complex ODE is of second order and we separate it into two coupled real equations, obtained by putting

$$f(\xi) = M(\xi) \exp(\mathrm{i}\chi(\xi)) \qquad M, \chi \in \mathbb{R}. \tag{1.3}$$

The equations obtained are linear in the phase and actually only involve the derivatives $\dot{\chi}$ and $\ddot{\chi}$. They are hence first-order equations for $\phi \equiv \dot{\chi}$. We solve for $\dot{\chi}$ from one equation and substitute into the other one.

In many cases (treated in II, and in [6, 7]) $\phi(\xi)$ is obtained directly in terms of the amplitude $M(\xi)$ and the substitution leads to a second-order ODE for $M(\xi)$.

In the remaining cases, treated below, the expression for $\phi(\xi)$ has the form

$$\phi(\xi) \equiv \dot{\chi}(\xi) = \beta(\xi) + \gamma(\xi) Y(\xi) \qquad Y(\xi) = \int M^2(\xi)\rho(\xi)\,\mathrm{d}\xi \tag{1.4}$$

where $\beta(\xi)$, $\gamma(\xi)$ and $\rho(\xi)$ are known. Substituting (1.4) back into the remaining equation, we obtain a third-order non-linear ODE for the auxiliary quantity $Y(\xi)$. Solving for $Y(\xi)$ we obtain both the phase χ and the amplitude M of $f(\xi)$ and hence a solution of the GNLSE (1.1).

As in the case of second-order ODE, we perform a Painlevé analysis of the obtained third-order ODE. This has some novel features, discussed below. Whenever the ODE passes the Painlevé test, we are able to integrate it, even though no complete classification of third-order Painlevé type equations exists (for some partial results, we refer to Bureau [8, 9]).

In § 2 we present the reduced ODE. Their singularity structure is discussed in § 3. Those equations that pass the test are integrated in § 4. The final section is devoted to conclusions.

2. The reduced equations

Since the reduction process, described in § 1, is completely standard, we skip all details and only present the results. Eight types of subgroups lead to third-order ODE (see I and II). All of them involve dilations and we are hence limited to purely cubic ($a_1 \neq 0$, $a_2 = 0$), or purely quintic ($a_1 = 0$, $a_2 \neq 0$) GNLSE of the form (1.1).

Let us run through the individual subgroups, identifying them by their Lie algebras. For notation we refer to I and II—and just recall that in the Galilei-similitude algebra the dilation generator is denoted d, rotations, space translations and proper Galilei transformations are denoted j_i, p_i and k_i ($i = 1, 2, 3$), respectively, whereas m denotes the operator generating a constant change of phase of the wavefunction and t corresponds to time translations.

The dilations act differently (see I) for the cases of the cubic and quintic GNLSE. In order to treat the two cases simultaneously, whenever possible, we introduce a parameter δ

$$\delta = \begin{cases} 1 & \text{for } a_1 \neq 0,\ a_2 = 0 \\ \frac{1}{2} & \text{for } a_1 = 0,\ a_2 \neq 0. \end{cases} \tag{2.1}$$

Throughout a and b denote real constants.

Subgroup 1. For $\{d + am, p_2, p_3\}$ we have:

$$\psi(\boldsymbol{r}, t) = f(\xi) t^{-\delta/2} \exp\left[-\mathrm{i}\left(a_0 t + \frac{a}{2} \ln t\right)\right] \qquad \xi = \frac{x}{\sqrt{t}} \tag{2.2a}$$

where f satisfies

$$\ddot{f} - \frac{\mathrm{i}}{2} \xi \dot{f} + \frac{a - \mathrm{i}\delta}{2} f = \begin{cases} a_1 |f|^2 f & \delta = 1 \\ a_2 |f|^4 f & \delta = \frac{1}{2}. \end{cases} \tag{2.2b}$$

Note that in II we used the variable $\xi = t/x^2$, instead of ξ as in (2.2a). The obtained ODE and the Painlevé analysis are somewhat simpler for the present (equivalent) choice.

Substituting (1.3) into (2.2b) we obtain

$$\ddot{M} - M\left(\dot{\chi} - \frac{\xi}{4}\right)^2 + \frac{\xi^2 + 8a}{16} M = \begin{cases} a_1 M^3 & \delta = 1 \\ a_2 M^5 & \delta = \frac{1}{2} \end{cases} \tag{2.2c}$$

$$M\ddot{\chi} + 2\dot{M}\dot{\chi} - \tfrac{1}{2}\xi\dot{M} - \tfrac{1}{2}\delta M = 0. \tag{2.2d}$$

The solution of (2.2d) is, for the cubic case,

$$\dot{\chi} = \frac{1}{4}\xi + \frac{1}{4M^2} Y(\xi) \qquad Y = \int M^2(\xi)\, \mathrm{d}\xi. \tag{2.2e}$$

Substituting $\dot{\chi}$ into (2.2c) we obtain the ODE for Y, namely, in the cubic case

$$2\dot{Y}\dddot{Y}-\ddot{Y}^2+\tfrac{1}{4}(\xi^2+8a)\dot{Y}^2-\tfrac{1}{4}Y^2-4a_1\dot{Y}^3=0. \qquad (2.2f)$$

In the quintic case ($a_1=0$, $a_2\neq 0$) the equation obtained is of second order, treated previously in II.

If we can solve the ODE (2.2f), then we obtain the amplitude M and phase χ of f as

$$M=\dot{Y}^{1/2} \qquad \chi=\frac{1}{8}\xi^2+\frac{1}{4}\int\frac{Y}{\dot{Y}}\,\mathrm{d}\xi+\chi_0 \qquad \chi_0=\text{constant}. \qquad (2.2g)$$

For the seven remaining subgroups we merely present a compendium of the most relevent formulae.

Subgroup 2. $\{d+am, p_2, k_3\}$:

$$\psi(\boldsymbol{r},t)=f(\xi)t^{-\delta/2}\exp\left[\mathrm{i}\left(\frac{z^2}{4t}-a_0t-\frac{a}{2}\ln t\right)\right] \qquad \xi=\frac{x}{\sqrt{t}} \qquad (2.3a)$$

$$2\dot{Y}\dddot{Y}-\ddot{Y}^2+\tfrac{1}{4}(\xi^2+8a)\dot{Y}^2-\begin{Bmatrix}\tfrac{1}{4}Y^2+4a_1\dot{Y}^3\\ Y^2+4a_2\dot{Y}^4\end{Bmatrix}=0 \qquad \begin{matrix}(2.3b)\\(2.3c)\end{matrix}$$

$$M=\dot{Y}^{1/2} \qquad \chi=\frac{1}{8}\xi^2+\frac{2\delta-3}{4}\int\frac{Y}{\dot{Y}}\,\mathrm{d}\xi+\chi_0. \qquad (2.3d)$$

Subgroup 3. $\{d+am, k_2, k_3\}$:

$$\psi(\boldsymbol{r},t)=f(\xi)t^{-\delta/2}\exp\left[\mathrm{i}\left(\frac{y^2+z^2}{4t}-a_0t-\frac{a}{2}\ln t\right)\right] \qquad \xi=\frac{x}{\sqrt{t}} \qquad (2.4a)$$

$$2\dot{Y}\dddot{Y}-\ddot{Y}^2+\tfrac{1}{4}(\xi^2+8a)\dot{Y}^2-\begin{Bmatrix}\tfrac{9}{4}Y^2+4a_1\dot{Y}^3\\ 4Y^2+4a_2\dot{Y}^4\end{Bmatrix}=0 \qquad \begin{matrix}(2.4b)\\(2.4c)\end{matrix}$$

$$M=\dot{Y}^{1/2} \qquad \chi=\frac{1}{8}\xi^2+\frac{2\delta-5}{4}\int\frac{Y}{\dot{Y}}\,\mathrm{d}\xi+\chi_0. \qquad (2.4d)$$

Subgroup 4. $\{d+am, p_3, j_3+bm\}$:

$$\psi(\boldsymbol{r},t)=f(\xi)t^{-1/4}\exp\left[-\mathrm{i}\left(a_0t+\frac{a}{2}\ln t-b\theta\right)\right] \qquad \xi=\left(\frac{x^2+y^2}{t}\right)^{1/2} \qquad (2.5a)$$

$$2\dot{Y}\dddot{Y}-\ddot{Y}^2+\frac{1}{4}\left(\xi^2+2a+\frac{1-4b^2}{\xi^2}\right)\dot{Y}^2-\frac{1}{4}Y^2-\frac{4a_2}{\xi^2}\dot{Y}^4=0 \qquad (2.5b)$$

$$M=\frac{1}{\sqrt{\xi}}\dot{Y}^{1/2} \qquad \chi=\frac{1}{8}\xi^2-\frac{1}{4}\int\frac{Y}{\dot{Y}}\,\mathrm{d}\xi+\chi_0. \qquad (2.5c)$$

The results (2.5) refer to the quintic GNLSE. For the cubic one ($a_1\neq 0$, $a_2=0$) this reduction provides a second-order ODE (II).

Subgroup 5. $\{d+am, k_3, j_3+bm\}$:

$$\psi(\boldsymbol{r}, t)=f(\xi)t^{-\delta/2}\exp\left[\mathrm{i}\left(\frac{z^2}{4t}+b\theta-a_0t+\frac{a}{2}\ln t\right)\right] \qquad \xi=\left(\frac{x^2+y^2}{t}\right)^{1/2} \tag{2.6a}$$

$$2\dot{Y}\dddot{Y}-\ddot{Y}^2+\left(\frac{1}{4}\xi^2+2a+\frac{1-4b^2}{\xi^2}\right)\dot{Y}^2-\left\{\begin{array}{l} Y^2+4a_1\dfrac{1}{\xi}\dot{Y}^3 \qquad (2.6b)\\[2ex] \dfrac{9}{4}Y^2+4a_2\dfrac{1}{\xi^2}\dot{Y}^4 \qquad (2.6c)\end{array}\right\}=0$$

$$M=\frac{1}{\sqrt{\xi}}\dot{Y}^{1/2} \qquad \chi=\frac{1}{8}\xi^2+\frac{\delta-2}{2}\int\frac{Y}{\dot{Y}}\,\mathrm{d}\xi+\chi_0. \tag{2.6d}$$

Subgroup 6. $\{d+am, j_1, j_2, j_3\}$:

$$\psi(\boldsymbol{r}, t)=f(\xi)t^{-\delta/2}\exp\left[-\mathrm{i}\left(a_0t+\frac{a}{2}\ln t\right)\right] \qquad \xi=\left(\frac{x^2+y^2+z^2}{t}\right)^{1/2} \tag{2.7a}$$

$$2\dot{Y}\dddot{Y}-\ddot{Y}^2+\frac{\xi^2+8a}{4}\dot{Y}^2-\left\{\begin{array}{l} \dfrac{1}{4}Y^2+4a_1\dfrac{1}{\xi^2}\dot{Y}^3 \qquad (2.7b)\\[2ex] Y^2+4a_2\dfrac{1}{\xi^4}\dot{Y}^4 \qquad (2.7c)\end{array}\right\}=0$$

$$M=\frac{1}{\xi}\dot{Y}^{1/2} \qquad \chi=\frac{1}{8}\xi^2+\frac{2\delta-3}{4}\int\frac{Y}{\dot{Y}}\,\mathrm{d}\xi+\chi_0. \tag{2.7d}$$

Subgroup 7. $\{d+bj_3+am, t, p_3\}$:

$$\psi(\boldsymbol{r}, t)=f(\xi)\rho^{-\delta}\exp[-\mathrm{i}(a_0t+a\ln\rho)] \qquad \xi=\theta+a\ln\rho \tag{2.8a}$$

$$\rho=(x^2+y^2)^{1/2} \qquad \theta=\tan^{-1}(y/x)$$

$$2\dot{Y}\dddot{Y}-\ddot{Y}^2-\left\{\begin{array}{l} \dfrac{16a^2}{(b^2+1)^4}Y^2+\dfrac{4(a^2-1)}{(b^2+1)^2}\dot{Y}^2+\dfrac{4a_1}{b^2+1}\exp\left(\dfrac{2b}{b^2+1}\xi\right)\dot{Y}^3 \qquad (2.8b)\\[2ex] \dfrac{4a^2}{(b^2+1)^4}Y^2+\dfrac{4a^2-1}{(b^2+1)^2}\dot{Y}^2+\dfrac{4a_2}{b^2+1}\exp\left(\dfrac{2b}{b^2+1}\xi\right)\dot{Y}^4 \qquad (2.8c)\end{array}\right\}=0$$

$$M=\exp\left(\frac{b\delta}{b^2+1}\xi\right)\dot{Y}^{1/2} \qquad \chi=\frac{ab}{b^2+1}\xi-\frac{2a\delta}{b^2+1}\int\frac{Y}{\dot{Y}}\,\mathrm{d}\xi+\chi_0. \tag{2.8d}$$

Subgroup 8. $\{d+am, t, j_3+bm\}$, $a\neq 0$:

$$\psi=f(\xi)\frac{1}{z}\exp[\mathrm{i}(b\theta-a_0t-a\ln z)] \qquad \xi=\frac{z}{(x^2+y^2)^{1/2}} \tag{2.9a}$$

$$2\dot{Y}\dddot{Y}-\ddot{Y}^2+\frac{8\xi}{1+\xi^2}\dot{Y}\ddot{Y}+\frac{1}{(1+\xi^2)^2}[(9-4b^2)\xi^2+6-4a^2-4b^2]\dot{Y}^2$$

$$-\frac{4a^2}{(1+\xi^2)^4}Y^2-4a_1\frac{1}{(1+\xi^2)^{1/2}}\dot{Y}^3=0 \tag{2.9b}$$

$$M=\xi(1+\xi^2)^{1/4}\dot{Y}^{1/2} \qquad \chi=\frac{a}{2}\ln\frac{\xi^2}{1+\xi^2}-a\int\frac{1}{(1+\xi^2)^2}\frac{Y}{\dot{Y}}\,\mathrm{d}\xi+\chi_0. \tag{2.9c}$$

For the quintic GNLSE this reduction gives a second-order ODE, treated previously in II.

The results of this section can be summed up quite succinctly. Thus, for the cubic GNLSE (1.1) with $a_1 \neq 0$, $a_2 = 0$ symmetry reduction by the eight subgroups of this section has led to six different third-order ODE. They can all be written in the form

$$2\dot{Y}\dddot{Y} - \ddot{Y}^2 + R(\xi)\dot{Y}\ddot{Y} + S(\xi)\dot{Y}^2 + T(\xi)Y^2 + U(\xi)\dot{Y}^3 = 0 \tag{2.10}$$

where the functions R, S, T and U are given in table 1 for each reduction.

Table 1. Coefficients in the reduced cubic GNLSE (2.10).

Number	Algebra	R	S	T	U
1, 2	$\{d + am, p_2, p_3\}$	0	$\dfrac{\xi^2 + 8a}{4}$	$-\dfrac{1}{4}$	$-4a_1$
	$\{d + am, p_2, k_3\}$				
3	$\{d + am, k_2, k_3\}$	0	$\dfrac{\xi^2 + 8a}{4}$	$-\dfrac{9}{4}$	$-4a_1$
4	$\{d + am, p_3, j_3 + bm\}$	—	—	—	—
5	$\{d + am, k_3, j_3 + bm\}$	0	$\frac{1}{4}\xi^2 + 2a + \dfrac{1 - 4b^2}{\xi^2}$	-1	$-\dfrac{4a_1}{\xi}$
6	$\{d + am, j_1, j_2, j_3\}$	0	$\dfrac{\xi^2 + 8a}{4}$	$-\dfrac{1}{4}$	$-\dfrac{4a_1}{\xi^2}$
7	$\{d + bj_3 + am, t, p_3\}$	0	$-\dfrac{4(a^2 - 1)}{(b^2+1)^2}$	$-\dfrac{16b^2}{(b^2+1)^4}$	$-\dfrac{4a_1}{b^2+1}\exp\dfrac{2b}{b^2+1}\xi$
8	$\{d + am, t, j_3 + bm\}$	$\dfrac{8\xi}{1 + \xi^2}$	$\dfrac{(9 - 4b^2)\xi^2 - 6 - 4a^2 - 4b^2}{(1+\xi^2)^2}$	$-\dfrac{4a^2}{(1+\xi^2)^4}$	$-\dfrac{4a_1}{(1+\xi^2)^{1/2}}$

Table 2. Coefficients in the reduced quintic GNLSE (2.11).

Number	Algebra	S	T	U
1	$\{d + am, p_2, p_3\}$	—	—	—
2	$\{d + am, p_2, k_3\}$	$\dfrac{\xi^2 + 8a}{4}$	-1	$-4a_2$
3	$\{d + am, k_2, k_3\}$	$\dfrac{\xi^2 + 8a}{4}$	-4	$-4a_2$
4	$\{d + am, p_3, j_3 + bm\}$	$\frac{1}{4}\xi^2 + \dfrac{a}{2} + \dfrac{1 - 4b^2}{4\xi^2}$	$-\dfrac{1}{4}$	$-\dfrac{4a_2}{\xi^2}$
5	$\{d + am, k_3, j_3 + bm\}$	$\frac{1}{4}\xi^2 + 2a + \dfrac{1 - 4b^2}{\xi^2}$	$-\dfrac{9}{4}$	$-\dfrac{4a_2}{\xi^2}$
6	$\{d + am, j_1, j_2, j_3\}$	$\dfrac{\xi^2 + 8a}{4}$	-1	$-\dfrac{4a_2}{\xi^4}$
7	$\{d + bj_3 + am, t, p_3\}$	$-\dfrac{4a^2 - 1}{(b^2+1)^2}$	$-\dfrac{4a^2}{(b^2+1)^2}$	$-\dfrac{4a_2}{b^2+1}\exp\dfrac{2b}{b^2+1}\xi$
8	$\{d + am, t, j_3 + bm\}$	—	—	—

For the quintic GNLSE (1.1) with $a_1 = 0$, $a_2 \neq 0$ the different reductions also lead to six different ODE, all of the form

$$2\dot{Y}\dddot{Y} - \ddot{Y}^2 + S(\xi)\dot{Y}^2 + T(\xi)Y^2 + U(\xi)\dot{Y}^4 = 0. \tag{2.11}$$

The functions S, T and U in this case are presented in table 2.

3. Singularity analysis of the third-order ordinary differential equations

In the previous section we obtained the reductions of the cubic and quintic NLSE to ODE which are *genuinely* of order three. The generic form of these equations is

$$2\dot{Y}\dddot{Y} - \ddot{Y}^2 + R(\xi)\dot{Y}\ddot{Y} + S(\xi)\dot{Y}^2 + T(\xi)Y^2 + U(\xi)\dot{Y}^n = 0 \tag{3.1}$$

with $n = 3$ for the cubic and $n = 4$ for the quintic equations. The functions R, S, T, U are not arbitrary; in fact they are given in table 1 or 2 for each of the obtained reductions. However, the singularity analysis can be performed in a quite general way without assuming any specific form for the parameter functions, at least up to the very last step of the calculations. One remark is in order at the outset. We assume that $T(\xi) \neq 0$ i.e. that $T(\xi_0)$ is non-zero at any generic point ξ_0. If this is not the case, i.e. if $T = 0$, then we may introduce a new variable $Z = \dot{Y}$ and reduce the ODE to second order. All these equations have been extensively studied in the previous paper (II) of the series. Therefore we will limit ourselves to genuinely third-order ODE and thus $T(\xi) \neq 0$.

The Painlevé analysis is performed on all the possible (singular) balances of the terms of the equation. Three such leading behaviours are possible.

(i) $2\dot{Y}\dddot{Y}$ exactly balances $\ddot{Y}^2$ and the other terms are of higher order. This means that $\dot{Y}$ behaves as z^2 (where $z = \xi - \xi_0$) and moreover that Y starts as z^3 (and not as $C + z^3$, $C \neq 0$, as Y^2 would then dominate). The resonances in this case are -3, -1 and 0 (the latter corresponding to the free coefficient of z^3 in Y). Thus, this leading behaviour (which moreover is a regular one) does not put any constraints on the parameter functions of equation (3.1). Still, we must stress here that the investigation of this leading behaviour is not a mere exercise in style. In fact, whenever the coefficient of the highest derivative in the equation may vanish (i.e. $\dot{Y} \to 0$ in our case) one must consider this leading behaviour, as it may introduce multivaluedness into the analytic structure of the solutions (as already shown in [10], see also the next paragraph).

(ii) $2\dot{Y}\dddot{Y} - \ddot{Y}^2$ balances TY^2. This case is again of the 'vanishing $\dot{Y}$' type. In fact, for this balance we have $Y = \alpha + \beta z^2 + \ldots$ where the constants α and β are related by

$$-4\beta^2 + T(\xi_0)\alpha^2 = 0$$

and thus $\dot{Y}$ behaves as $2\beta z$. The zero of $\dot{Y}$ can, in principle, be non-regular. The resonances in this case turn out to be -1, 0 and 1 in the expansion of $\dot{Y}$. They correspond respectively to the location ξ_0 of the zero, to the value of β (which determines α) and to the free coefficient of z^2 in $\dot{Y}$. In the expansion of Y, the resonances would be 0, 1 and 3. One genuine compatibility condition enters at first order in the $\dot{Y}$ expansion and it is

$$R(\xi) = -\partial \ln(T(\xi))/\partial\xi. \tag{3.2}$$

For the *most general* equation of type (3.1), this leading behaviour is indeed singular. Indeed, logarithms appear in the expansion, though it does not start as a pole but rather as an ordinary point with vanishing first derivative. However, it so happens that condition (3.2) is satisfied for *all* the reductions of both the cubic and the quintic equations. This is no mere coincidence. The leading behaviour analysed here can in fact be traced back to the original, non-reduced, NLS equation. The resonance condition can then be investigated directly for the partial differential equation [11]. We will not reproduce this lengthy calculation here, but it suffices to say that the compatibility condition is indeed satisfied by the non-reduced NLS equation

(iii) $2\dot{Y}\dddot{Y}-\ddot{Y}^2$ balances the 'non-bilinear' term $\dot{Y}^n$ $(n=3,4)$.

Let us first treat the quintic case $(n=4)$. We readily find that $\dot{Y}$ must behave as z^{-1} and thus Y behaves as $\ln z$. This logarithmic behaviour of Y, *per se*, would not have been a problem, were it not for the presence of the TY^2 term in the equation (recall that T never vanishes for the equations we consider). Through this term, logarithms enter into the expansion of Y at all orders. Thus, no reduction of the quintic NLS equation to a third-order equation possesses the Painlevé property.

We now turn to the cubic case. Here $\dot{Y}$ behaves as z^{-2}, and thus Y diverges as z^{-1}. The resonances here are -1, 1 and 4. Two resonance conditions must be satisfied in this case. At the resonance 1, we obtain

$$R(\xi)=-4\,\partial\ln(U(\xi))/\partial\xi. \tag{3.3}$$

This first condition is already a most powerful one. In fact, a look at table 1 suffices to convince oneself that only cases $\{1,2\}$ and $\{3\}$ satisfy equation (3.3) and thus have some hope of having the Painlevé property. (Note that case $\{7\}$ with $b=0$ would also satisfy equation (3.3) but then we have $T=0$ and thus equation (3.1) reduces to second order, as treated in II.)

Let us now turn to the remaining resonance 4. The compatibility condition at this resonance is a rather lengthy one, involving the parameter functions R, S, T, U and their derivatives. However, for the two surviving cases, as $R=0$ and U is a constant, it assumes a most simple form, namely

$$\partial^2 S(\xi)/\partial\xi^2+2T(\xi)=0 \tag{3.4}$$

which shows that case $\{1,2\}$ does pass the Painlevé test while case $\{3\}$ does not. Thus to summarise the results of our singularity analysis for the third-order reductions of the NLS equation, we have found the following.

(i) None of the reductions of the quintic equation passes the Painlevé test.

(ii) Among the reductions of the cubic equation, only

$$2\dot{Y}\dddot{Y}-\ddot{Y}^2+\frac{\xi^2+8a}{4}\dot{Y}^2-\frac{1}{4}Y^2-4a_1\dot{Y}^3=0 \tag{3.5}$$

is of Painlevé type. In fact this equation results from two different reductions and is the same as would have resulted from a reduction of the $(1+1)$-dimensional cubic NLS equation. The latter being integrable by the IST method, its reduction should have the Painlevé property, according to the ARS conjecture [5], and it does indeed. Its integration in terms of Painlevé transcendents is given in the next section.

It is worth mentioning that the singularity analysis of this section is more elaborate than the usual 'Painlevé tests'. In particular it goes beyond the cases handled by the MACSYMA program of [12]. Since the highest derivative $\dddot{Y}$ is not isolated in (3.1) (it is multiplied by $\dot{Y}$) it is also necessary to investigate power expansions of Y that start out as z^α, where α may be zero or a positive integer. The program of [12] is restricted to values of α that are negative integers.

4. Integration of the Painlevé type equation

Equation (3.5) was shown to pass the Painlevé test in the previous section. We now proceed to integrate it in terms of known functions, namely the Painlevé transcendent P_{IV}. The procedure we use is a reasonably general one, that can be applied to reducing the order of a Painlevé type equation. It will fail whenever the equation describes a new transcendent, e.g. the six Painlevé transcendents in the case of second-order equations of the form $\ddot{y} = F(y, \dot{y}, t)$, where F is rational in y and $\dot{y}$.

We first look for a first integral of equation (3.5) in the form

$$\ddot{Y} = M(\xi, Y, \dot{Y}) + KN(\xi, Y, \dot{Y}) \tag{4.1}$$

where M and N are rational functions of Y and $\dot{Y}$, whereas K is a constant. Differentiating with respect to ξ, eliminating K from the resulting third-order equation (using (4.1)) and comparing with (3.5) we see that no such first integral exists.

Next, we make a more general ansatz, namely that the first integral of (3.5) has the form

$$\ddot{Y}^2 + F(Y, \dot{Y}, \xi)\ddot{Y} = A(\xi, Y, \dot{Y}) + KB(\xi, Y, \dot{Y}) \tag{4.2}$$

where K is again a constant and F, A and B are rational in Y and $\dot{Y}$. Differentiating (4.2) and comparing with (3.5) we find $F = 0$ and obtain A and B as simple polynomials. The result is that (3.5) has a first integral of the form

$$\ddot{Y}^2 = -\tfrac{1}{4}(Y - \dot{Y}\xi)^2 - 2a\dot{Y}^2 + 2a_1\dot{Y}^3 + C\dot{Y} \tag{4.3}$$

where C is an integration constant. The second-order equation (4.3) belongs to the class of equations of the form

$$A(\dot{y}, y, x)\ddot{y}^2 + B(\dot{y}, y, x)\ddot{y} + C(\dot{y}, y, x) = 0 \tag{4.4}$$

studied by Bureau [9].

More specifically, we put

$$Y(\xi) = \frac{1}{a_1}\left(-kw(x) + \frac{a}{3k}x\right) \qquad x = k\xi \qquad k^4 = -\tfrac{1}{16}. \tag{4.5}$$

The function $w(x)$ then satisfies

$$\ddot{w}^2 - 4(w - \dot{w}x)^2 + 2\dot{w}^2 + \frac{16}{3}(2a^2 - 3Ca_1)\dot{w} - \frac{16}{27k^2}(4a^3 - 9Caa_1) = 0. \tag{4.6}$$

Equation (4.6) coincides with equation (22.15) of Bureau [9] if one chooses two constants α and β (a and b of [9]) to satisfy

$$\begin{aligned} &-2(\alpha+1)^2 + 3\beta = 4(2a^2 - 3Ca_1) \\ &(\alpha+1)[2(\alpha+1)^2 + 9\beta] = -(a/k^2)(4a^2 - 9Ca_1). \end{aligned} \tag{4.7}$$

Following Bureau [9], we express the solution of (4.6), and hence of (4.3), in terms of the Painlevé transcendent

$$y(x) \equiv P_{IV}(\alpha, \beta, C_1, C_2, x) \tag{4.8}$$

where C_1 and C_2 are two integration constants (the third one C is absorbed in α and β).

The solution of (4.3) is finally given by

$$Y(\xi)=-\frac{k}{a_1}\left[\frac{\dot{y}^2}{4y}-\frac{y^3}{4}-xy^2+(1-x^2+\alpha)y+\frac{\beta}{2y}+\frac{2}{3}(\alpha+1)x-\frac{ax}{3k^2}\right] \tag{4.9}$$

where $y(x)$ of (4.8) satisfies the P_{IV} equation

$$\ddot{y}=\frac{\dot{y}^2}{2y}+\frac{3}{2}y^3+4xy^2+2(x^2-\alpha)y+\frac{\beta}{y}. \tag{4.10}$$

We thus obtain two types of solutions of the cubic GNLSE (1.1) with $a_2=0$. The first is given by (2.2a) with $\delta=1$, $f(\xi)=M(\xi)\exp(\mathrm{i}\chi(\xi))$, $\xi=xt^{-1/2}$, and $M(\xi)$ and $\chi(\xi)$ given in terms of $Y(\xi)$ in (2.2g). The second type of solution is given by (2.3a), with $\delta=1$, $f(\xi)=M(\xi)\exp(\mathrm{i}\chi(\xi))$, $\xi=xt^{-1/2}$, and $M(\xi)$ and $\chi(\xi)$ given in terms of $Y(\xi)$ in (2.3d).

The results for the reduction (2.2) using the subalgebra $\{d+am, p_2, p_3\}$ agree with those of Boiti and Pempinelli [13], but are somewhat more general, since [13] deals only with the case $a=0$.

5. Conclusions

This paper, taken together with I and II, as well as [6, 7] completes the task of finding all group invariant solutions of the GNLSE (1.1) that are of Painlevé type. We have seen that Painlevé type ODE of third order are quite rare among the equations obtained from (1.1) by symmetry reduction. Indeed, they never occur for $a_2\neq 0$ and only in two cases out of seven for $a_2=0$, $a_1\neq 0$.

To obtain further solutions we must either give up the group invariance condition, or find methods for solving the obtained ODE that do not have the Painlevé property. Work on partially invariant solutions (as defined by Ovsiannikov [14]) is in progress. We do not know, at this stage, how to solve the non-Painlevé type ODE of this paper and of II analytically. However, their qualitative behaviour can be investigated, using known methods, and numerical solutions could easily be calculated.

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